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DISCUSSIONS ON THE TIME DOMAIN INTEGRAL REPRESENTATIONS FOR ELASTODYNAMICS ANALYSIS

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Abstract: *This article describes three possible ways to derive time domain boundary integral representations of displacements for elastodynamics. The subject is quite simple, but this discussion will point out possible difficulties found when using those formulations to deal with practical applications related with the boundary element method. The discussion can give us recommendations to select properly the convenient integral representation to deal with this kind of problem and may open the possibility of deriving simplified schemes. For instance, the proper way to take into account initial conditions applied to the body is an interesting point to discuss and can illustrate the main differences between the mentioned boundary integral representation expressions, their singularities and possible numerical difficulties to be faced.*

Key words: Elastodynamics, integral representations, Boundary element method.

1. INTRODUCTION

The boundary element community has already seen many interesting works on elastodynamics. The complexity of the subject has attracted attention of many researchers around the world. Complete and useful reviews on this subject have been published by Beskos (1987,1997). In particular, we will focus our discussion on Time Domain Boundary Element Method (TDBEM) for elastodynamic problems. As this is a very well known subject, the authors intend only to discuss the possible ways to derive the time domain boundary integral representation for elastodynamics, pointing out the main differences between them, and their difficulties to solve problems defined by bodies with large boundary volume relation. How to deal with initial conditions is another interesting point to discuss.

2. THE ELASTODYNAMIC PROBLEM

The governing differential equation of linear elastodynamic equilibrium (i.e. the Navier-Cauchy equation) is given by,

$$\left(C_1^2 - C_2^2\right)u_{j,ji} + C_2^2 u_{i,jj} + b_i / \rho = \ddot{u}_i \tag{1}$$

where b_i and u_i are body force and displacement fields, respectively, ρ stands for the medium density, while C_1 and C_2 represent longitudinal and shear wave propagation velocities, respectively.

Equation (1) can also be written in terms of stresses, σ_{ij}, as follows,

$$\sigma_{ij,i} + b_j = \rho \frac{\partial^2 u_j}{\partial t^2} = \rho \ddot{u}_j \tag{2}$$

Considering a body defined in the domain Ω with boundary Γ, the following boundary conditions along time must be imposed:

$$\begin{cases} u_i(x,t) = \bar{u}_i(x,t) & x \in \Gamma_1 \\ p_i(x,t) = \bar{p}_i(x,t) & x \in \Gamma_2 \end{cases} \tag{3}$$

where $p_i(x,t)$ represents boundary tractions obtained from the stress field using the Cauchy's formula, being the whole boundary given by $\Gamma = \Gamma_1 + \Gamma_2$.

As usual the initial conditions are given by:

$$\begin{cases} u_i(x,t_0) = u_{i0}(x) \\ \dot{u}_i(x,t_0) = v_{i0}(x) \end{cases} \qquad x \in \Omega \tag{4}$$

3. GRAFFIS RECIPROCAL THEOREM

Using the weighting residual technique together with equation (2), one can easily achieve the Graffi's reciprocal theorem, as follows,

$$\begin{aligned} &\int_{t_0}^{t}\int_{\Omega} u_j(\tau)\tilde{b}_j(t-\tau)d\Omega d\tau + \int_{t_0}^{t}\int_{\Gamma} u_j(\tau)\tilde{p}_j(t-\tau)d\Gamma d\tau = \int_{t_0}^{t}\int_{\Gamma} p_j(\tau)\tilde{u}_j(t-\tau)d\Gamma d\tau + \\ &+ \int_{t_0}^{t}\int_{\Omega} b_j(\tau)\tilde{u}_j(t-\tau)d\Omega d\tau + \int_{t_0}^{t}\int_{\Omega} \sigma_{ij}^{a}(\tau)\tilde{\varepsilon}_{ij}(t-\tau)d\Omega d\tau + \\ &+ \int_{\Omega} \rho\{\dot{u}_j(\tau)\tilde{u}_j(t-\tau) - u_j(\tau)\dot{\tilde{u}}_j(t-\tau)\}\Big|_{t_0}^{t} d\Omega \end{aligned} \tag{5}$$

where $\tilde{u}_i(\tau), \tilde{\sigma}_{ij}(\tau)$ are weighting fields, $u_i(\tau)$ and $\sigma_{ij}(\tau)$ represent the actual solid displacement and stress fields, while $\sigma^a_{ij}(\tau)$ stands for the initial stress state. Particular attention must be given to the last term in equation (5), representing the initial conditions.

4. BOUNDARY INTEGRAL REPRESENTATION FOR DISPLACEMENT

In order to derive the boundary integral representation for displacements from equation (5), one has to assume the following distribution for the weighting field body force (Wheeler & Sternberg, 1968).

$$b^*_{kj} = \delta(s-q)\delta_{kj} f(\tau) \tag{6}$$

where $\delta(s-q)$ is the Dirac delta in space, ‘s’ and ‘q’ represent the source and field points, δ_{kj} stands for the Kronecker delta, while $f(\tau)$ gives the time behaviour of this set of loads.

For this particular body force distribution, the corresponding weighting field $\{\tilde{u}_{ki}(\tau), \tilde{\sigma}_{kij}(\tau)\}$ represents the general Stokes' state. Replacing that state and expression (6) into equation (5) leads to the following integral representation:

$$\begin{aligned}
&C_{ki}(Q,s)\int_{t_0}^{t} u_i(s,\tau) f(t-\tau) d\tau + \int_{t_0}^{t}\int_{\Gamma} u_j(\tau)\tilde{p}_{kj}(Q,t;s|f(\tau)) d\Gamma d\tau = \\
&\int_{t_0}^{t}\int_{\Gamma} p_j(\tau)\tilde{u}_{kj}(Q,t,s|f(\tau)) d\Gamma d\tau + \int_{t_0}^{t}\int_{\Omega} b_j(\tau)\tilde{u}_{kj}(Q,t,s|f(\tau)) d\Omega d\tau + \\
&\int_{t_0}^{t}\int_{\Omega} \sigma^a_{ij}(\tau)\tilde{\varepsilon}_{kij}(Q,t,s|f(\tau)) d\Omega d\tau + \int_{\Omega} \rho\{\dot{u}_j(\tau)\tilde{u}_{kj}(Q,t,s|f(\tau)) - u_j(\tau)\tilde{\dot{u}}_{kj}(Q,t,s|f(\tau))\}\Big|_{t_0}^{t} d\Omega
\end{aligned} \tag{7}$$

From equation (7), one can follow several alternatives to write a displacement integral representation to obtain conveniently TDBEM. For instance, in Kobayashi (1987), Mansur (1988) and Manolis (1986), one can see an elegant and straightforward way to obtain the desired representation. This approach consists of replacing, in equations (7) and (6), the time function $f(\tau)$ by the Dirac's delta distribution $\delta(\tau)$ resulting:

$$\begin{aligned}
&C_{ki}(Q,s) u_i(s,t) = \int_{t_0}^{t}\int_{\Gamma} u^*_{ki}(Q,t;s,\tau) p_i(Q,\tau) d\Gamma d\tau + \\
&-\int_{t_0}^{t}\int_{\Gamma} u_i(Q,\tau) p^*_{ki}(Q,t;s,\tau) d\Gamma d\tau + \int_{t_0}^{t}\int_{\Omega} u^*_{ki}(q,t;s,\tau) b_i(q,\tau) d\Omega d\tau \\
&\int_{t_0}^{t}\int_{\Omega} \sigma^a_{ij}(\tau)\varepsilon^*_{kij}(Q,t;s,\tau) d\Omega d\tau + \int_{\Omega} \rho\{\dot{u}_j(\tau) u^*_{kj}(Q,t;s,\tau) - u_j(\tau)\dot{u}^*_{kj}(Q,t;s,\tau)\}\Big|_{t_0}^{t} d\Omega
\end{aligned} \tag{8}$$

where the fields $u^*_{ki}(\tau)$ and $\sigma^*_{kij}(\tau)$ represent the Dirac's delta type fundamental solution given in terms of displacements and stresses respectively.

Equation (8) is called here the *First Dirac's Delta Displacement Integral representation* or simply, *Loves' Identity* (LI).

An alternative to this way followed to derive the displacement integral representation can be found in Karabalis & Beskos (1987). It consists of applying again Dirac's function, in expression (7), to achieve first equation (8). Then, after deriving that integral representation, equation (8), they took a well-known Dirac's delta fundamental solution property (Eringen & Suhubi, 1974), i.e., the convolution between the Dirac's delta fundamental solution and any order function gives a Stokes' state, exhibiting the impulse distribution governed by the adopted function. This property transforms equation (8) into the more elegant form given by:

$$C_{ki}(Q,s)u_i(s,t) = \int_\Gamma u^0_{ki}(Q,t,s|p_i(Q,t))d\Gamma - \int_\Gamma p^0_{ki}(Q,t,s|u_i(Q,t))d\Gamma$$

$$+\int_\Omega u^0_{ki}(Q,t,s|b_i(Q,t))d\Omega + \int_\Omega \varepsilon^0_{kij}(Q,t,s|\sigma^a_{ij}(Q,t))d\Omega +$$

$$\int_\Omega \rho\{\dot{u}_j(\tau)u^*_{kj}(Q,t;s,\tau) - u_j(\tau)\dot{u}^*_{kj}(Q,t;s,\tau)\}\Big|^t_{t0} d\Omega \qquad (9)$$

where the field $\left\{u^0_{ki}(\tau), \sigma^0_{kij}(\tau)\right\}$ is the Stokes' state related to the behaviour of the unknown variables $\{u,p\}$ and the field $\left\{u^*_{ki}(\tau), \sigma^*_{kij}(\tau)\right\}$ is the Dirac's delta fundamental solution, as stated in equation (8).

Note that the Dirac's fundamental solution is still present in the initial condition integral term. As the same weighting function has been used to achieve both equations (8) and (9), the latter is named here the *Second Dirac's Delta Displacement integral representation***,** or simply the *Compact Loves' Identity* (CLI).

Equations (8) and (9) were obtained using the same weighting function, therefore they exhibit the same singularities. They differ basically on the convolution timing. The convolution has already been performed in equation (9), while it must be still performed in equation (8).

More recently, Coda & Venturini (1995,1996) have been working on this subject and have verified (not reported in the literature yet) that both procedures described above give precise results only for problems exhibiting very small boundary/volume ratio. For problems exhibiting complex boundary, stable numerical solutions are quite difficult. The way found by the authors to improve the TDBEM stability was obtained by reducing the singularities of the kernels. Instead of assuming an instantaneously concentrated impulse, $b^*_{kj} = \delta(s-q)\delta_{kj}\delta(\tau)$, they started by replacing $\delta(\tau)$ by the Heaviside distribution. The weighting function adopted to derive the integral representation is therefore given by a concentrated unit load assumed distributed over a time interval (Δt), as follows:

$$b^*_{ki} = \left[H(\tau) - H(\tau - \Delta t)\right]\delta(q-s)\delta_{ki} / \Delta t \qquad (10)$$

One can also choose smoother distributions depending on the singularity reduction desired. Another possible formula to represent b^*_{kj} is given by making:

$$f(\tau)=\frac{g(\tau)-g(\tau-R_d*R_t)}{R_d} \tag{11a}$$

with

$$g(\tau)=\left[56\left(\frac{\tau}{R_t}-\frac{1}{2}\right)^7-36\left(\frac{\tau}{R_t}-\frac{1}{2}\right)^5+\frac{11}{2}\left(\frac{\tau}{R_t}-\frac{1}{2}\right)^3+\frac{\tau}{R_t}\right]\frac{[H(\tau)-H(\tau-R_t)]}{R_t}+\frac{H(\tau-R_t)}{R_t} \tag{11b}$$

where R_t is a sub-element of Δt to define the time that the load function requires to reach its maximum value and R_d represents the duration the load function remains at its maximum value ($R_t*R_d=\Delta t-2R_t$).

Adopting the fundamental solution derived by choosing b_{kj}^* given by equation (10) or (11) the Graffi's theorem is modified to give:

$$C_{ki}(Q,s)\hat{u}_i(s,\Delta t)+\int_{t0}^{t}\int_{\Gamma}u_j(\tau)\bar{p}_{kj}(Q,t;s|f(\tau))d\Gamma d\tau=\int_{t0}^{t}\int_{\Gamma}p_j(\tau)\bar{u}_{kj}(Q,t,s|f(\tau))d\Gamma d\tau+$$

$$+\int_{t0}^{t}\int_{\Omega}b_j(\tau)\bar{u}_{kj}(Q,t,s|f(\tau))d\Omega d\tau+\int_{t0}^{t}\int_{\Omega}\sigma_{ij}^{a}(\tau)\bar{\varepsilon}_{kij}(Q,t,s|f(\tau))d\Omega d\tau+$$

$$+\int_{\Omega}\rho\{\dot{u}_j(\tau)\bar{u}_{kj}(Q,t,s|f(\tau))-u_j(\tau)\bar{\dot{u}}_{kj}(Q,t,s|f(\tau))\}\Big|_{t0}^{t}d\Omega \tag{12}$$

where $\hat{u}_i(s,\Delta t)=\int_{t-\Delta t}^{t}u_i(s,\tau)f(t-\tau)d\tau$ is understood as an average displacement value over the final time step. When the time approximation is assumed, this value becomes the time parametric displacements. The field $\{\bar{u}_{ki}(\tau),\bar{\sigma}_{kij}(\tau)\}$ is the Stokes' state related to the time load function given in equation (10) or (11).

The state achieved above assuming the impulse distributed along a time is named here smooth fundamental solution. Note that this smooth fundamental solution is present in the initial condition term in equation (12).

As the Dirac's delta fundamental solution has not been used to achieve expression (12), this representation is named here the *Smooth Displacement Integral Representation***,** or simply *Smooth Loves' Identity* (SLI).

5. DISCUSSIONS

We intend to make comments about the integral representations given by equations (8), (9) and (12). Although the two first representations have been many times adopted in the literature to build TDBEM, it is not possible to find good results for dynamic problem analysis charaterized by exhibiting complex boundary geometry. Practically, they have been applied to half space problems only, where no important wave reflections are present. By the experience we have in solving many elastodynamic problems, TDBEM algebraic relations obtained by using equation (12) give much more stable results. They allow us to face problems that with more complex boundaries. It must be said that improvements on this solution regarding stability are still required, but in comparison with the solution given by the old formulations, we have obtained rather better results.

Recently, we discussed with Prof. D. Beskos (1998) our formulation. He believes that our propose is the same one that he published together with Karabalis (1984). They have used equation (9), i.e. they have used the *Compact Loves' Identity* (CLI). Then, they approached the tractions along the boundary using a Heaviside distribution. Their expression (13), given at page 78 of reference (Beskos, 1984), is, in fact, similar to expression (10) given here. Their expression is an approximation of tractions, while our formula is a domain load approximation. It is not possible reducing singularity by smoothing traction representation only. Using CLI or LI, all problems regarding stability remain.

The kernel singularity reduction proposed here does not depend on the approximation assumed for tractions or displacements after obtaining the integral representation. It is true that singularity reduction can be reached by using higher order time approximations, but it will be computationally more expensive and does not solve the initial condition problem.

Another clear difference between those representations is related with the capability of dealing with initial conditions. One can observe that the integral representations (3D) derived by using Dirac's delta fundamental solution, LI or CLI, exhibit domain integral terms in $\delta(\ )$ and $\dot{\delta}(\ )$ to take into account initial velocity and displacement fields, respectively. Prof. Beskos does not agree with that although in (Beskos, 1984) the initial velocity and displacement terms are weighted by $u^*_{ki}(x,t,\xi)$ and $\dot{u}^*_{ki}(x,t,\xi)$, therefore written in terms of $\delta(\ )$ and $\dot{\delta}(\ )$, respectively. It must be said that only one single paper has been published using equation (8) or (9) to face initial condition (Antes, h. & Steinfeld, 1992). Even so, the authors have discussed the self-weight domain integral term only, for which they have been forced to define spherical surfaces inside the domain. The problem has been discussed by Sladek (1992) following a more complex mathematical approach, but without presenting numerical results.

On the contrary, for the smooth integral representation, one can directly apply expression (10) to take into account initial velocity fields, as it has been shown previously (Coda & Venturini, 1995a). Initial displacement fields can be easily taken into account by approaching the load function using a piecewise linear approximation. Obtaining the stress integral representation is another difficult task for the Dirac's delta type integral representations. Again, the smooth representation can provide a convenient way to obtain that representation.

Another difference between smooth and Dirac's delta type fundamental solutions, expressions (8), (9) and (12), is regarding their behaviours when dealing with constant time approximation. The difference between expressions (8) and (9) is clearly given by the timing of their convolutions. The convolution in equation (8) is performed assuming a simple time approximation for the unknown values, while for expression (9), the unknown behaviours along time steps are understood as time distributed loads adopted to build the kernels after the convolution, and thus no more convolution is required. Moreover, constant time approximation for displacements can not be adopted in equation (8); by using this approximation the first time derivative of Dirac's delta distribution, present in the fundamental solution, will be eliminated during the convolutional process. This justify why at least linear approximation for displacements is required to achieve TDBEM algebraic representations based on LI integral equations.

If displacements are assumed piece-wisely constant, distributed impulses over time steps to generate the kernels of expression (9), Dirac's delta distribution will be necessarily found inside the corresponding kernel. Thus, curved surfaces inside the body must be provided to perform space integrals required to take into account jump conditions and to obtain correct values. In the first work published on this approach, CLI, the authors have not worked out on those terms (Karabalis & Beskos, 1987). Their approach did not disturb the results because they solved only half-space problems where computing H matrix is not required.

The main difference between SLI and the other previous formulation is regarding the convolution timing. SLI exhibits a Stokes' state convoluted in time without taking into account its time behaviour that will be defined later. It is very easy to verify that applying the weighting function, given in equation (10), the TDBEM based on SLI can deal with constant time approximations for both displacements and tractions. This scheme has already proved to give excellent results when compared with other works in which the algebraic representations are based on LI or CLI (Coda, 1993) and (Coda & Venturini, 1995b, 1996a, 1996b).

Nowadays, BEM researchers are worried about the stability of TDBEM (Siebrits & Peirce, 1997 and Risos & Karabalis, 1997) trying to find proper way to improve the time responses. However, this effort has been made on the LI or CLI integral representations, neglecting therefore the difficulties regarding the initial conditions. In order to achieve further and quicker progress on TDBEM approaches, efforts must be concentrated to stabilise the numerical responses. The *Smooth Love's Identity*, SLI has already proved to give more stable results, to make easier the initial condition treatment and to be more convenient to derive other integral representations by differentiating the displacement equation. Thus, this integral representation is strongly recommended for new TDBEM elastodynamic developments.

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